UNIVERSITY OF SWAZILAND

FACULTY OF SCIENCE

DEPARTMENT OF PHYSICS

SUPPLEMENTARY EXAMINATION 2007/2008

TITLE OF PAPER : LINEAR ALGEBRA AND VECTOR CALCULUS

COURSE NUMBER : E372

TIME ALLOWED : THREE HOURS

INSTRUCTIONS : ANSWER ANY FOUR OUT OF FIVE QUESTIONS. EACH QUESTION CARRIES 25 MARKS.

MARKS FOR DIFFERENT SECTIONS ARE SHOWN IN THE RIGHT-HAND MARGIN.

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E372 Linear Algebra and Vector Calculus

Question one

Given the following matrix equation A X = b where

$$A = \begin{pmatrix} 7 & 3 & -2 \\ 3 & -8 & 4 \\ 1 & 4 & -5 \end{pmatrix} \quad , \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad and \quad b = \begin{pmatrix} 33 \\ 38 \\ -13 \end{pmatrix}$$

(a) use the "linsolve" command to find the solution of X , **(3 marks)**

(b) use the Gauss elimination method to find the solution of X, **(6 marks)**

(c) use the Crammer's rule to find the solution of X , **(6 marks)**

(d) (i) find A^{-1} , **(8 marks)**

(ii) use A^{-1} obtained in (d) (i) to find the solution of X . **(2 marks)**

Question two

Given the following differential equations for a coupled oscillator system as

$$\begin{cases} \dfrac{d^2 x_1(t)}{dt^2} = -15\, x_1(t) + 6\, x_2(t) \\ \dfrac{d^2 x_2(t)}{dt^2} = 8\, x_1(t) - 28\, x_2(t) \end{cases}$$

(a) set $x_1(t) = X_1\, e^{i\omega t}$ *and* $x_2(t) = X_2\, e^{i\omega t}$, deduce the following matrix equation $A\, X = -\,\omega^2\, X$ where

$$A = \begin{pmatrix} -15 & 6 \\ 8 & -28 \end{pmatrix} \quad and \quad X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} ,$$ **(4 marks)**

(b) (i) find the eigen frequencies of ω , **(4 marks)**

(ii) find the eigen vectors of X , **(4 marks)**

(c) (i) write down the general solutions of $x_1(t)$ *and* $x_2(t)$ in terms of the eigen frequencies and eigen vectors obtained in (b) , **(4 marks)**

(ii) if initial conditions are given as

$$x_1(0) = 5 \ , \quad x_2(0) = 0 \ , \quad \left.\frac{d\, x_1(t)}{dt}\right|_{t=0} = 0 \quad and \quad \left.\frac{d\, x_2(t)}{dt}\right|_{t=0} = 0 \ ,$$

find the specific solutions of $x_1(t)$ *and* $x_2(t)$. Plot both $x_1(t)$ *and* $x_2(t)$ for $t = 0$ *to* 5 and show them in a single display. **(9 marks)**

Question three

(a) Given a scalar function $f = 12\,e^{-2x} + 5\,x\,y^3 - y^2\,z^2$,

(i) find the value of $\vec{\nabla} f$ at the point P : (-1 , -3 , 4) , **(3 marks)**

(ii) find the directional derivative of f at the point P : (-1 , -3 , 4) along the direction of [- 1 , 6 , - 2] . **(4 marks)**

(b) Given a vector field $F = [\,15\,y\,e^{-3x}, -5\,e^{-3x}, 24\,z^2\,]$, find the value of the line integral of F from the point P_1 : (0 , 0 , 0) to the point P_2 : (2 , 8 , 0) along a line path of L , i.e., $\int_{P_1, L}^{P_2} \vec{F} \bullet d\vec{l}$,

(i) if L : z = 0 and y = 4 x , **(7 marks)**

(ii) if L : z = 0 and $y = x^3$, **(7 marks)**

(iii) is the given F a conservative vector field ? If so, then find its associated scalar potential. **(4 marks)**

Question four

(a) Given a vector field $F = [\, 5\, x^2\, z \,, -\, 2\, z^2\, x \,, 3\, x\, y\, z \,]$, find the value of the surface integral of F , i.e., $\iint_S \vec{F} \bullet d\,\vec{s}$, if

$S \,:\, x^2 + y^2 = 16 \;,\; x > 0 \;,\; y > 0$ and $0 \le z \le 7$. **(10 marks)**

(b) Given a vector field $G = [\, 5\, x^2 \,, -\, 7\, x\, y \,, 3\, y\, e^{-z} \,]$, utilize the divergence theorem to find the closed surface integral of G , i.e., $\oiint_S \vec{G} \bullet d\,\vec{s}$ if S is the closed surface enclosing the following volume

$V \,:\, 0 \le x \le 3 \;,\; 0 \le y \le 8$ and $0 \le z \le 5$. **(10 marks)**

(c) For any scalar field f(x,y,z) , show that

$\vec{\nabla} \times \left(\vec{\nabla} f \right) \equiv 0$. **(5 marks)**

Question five

(a) Given the following partial differential equation as

$$\frac{y}{x}\frac{\partial f(x,y)}{\partial x}+\frac{x}{y}\frac{\partial f(x,y)}{\partial y}=0$$

set $f(x,y)=F(x)\,G(y)$ and utilize the separation of variable scheme to break the above partitial differential equation into two ordinary differential equation,

(6 marks)

(b) Given a periodical function f(x) of period 18 , i.e., f(x) = f(x+18) , as

$$f(x)=\begin{cases} 5 & if \quad 0\le x\le 13 \\ -x+18 & if \quad 13\le x\le 18 \end{cases}$$

find its Fourier series representation up to a_{10} *and* b_{10} terms for cosine and sine series respectively. Plot the truncated series representation of f(x) for $x=0$ *to* 18 . **(10 marks)**

(c) (i) Show that the left hand side of equation below represents the Fourier integral of the right hand side function.

$$\int_0^\infty \frac{\omega\sin(x\,\omega)}{(\omega^2+25)}\,d\,\omega=\begin{cases} -\frac{\pi}{2}e^{5x} & if \quad x\le 0 \\ \frac{\pi}{2}e^{-5x} & if \quad x\ge 0 \end{cases}$$

(7 marks)

(ii) find the value of the left hand side integral if x = - 0.8 . **(2 marks)**