UNIVERSITY OF SWAZILAND

FACULTY OF SCIENCE

DEPARTMENT OF ELECTRONIC ENGINEERING

MAIN EXAMINATION 2007/2008

TITLE OF PAPER : COMPLEX VARIABLES

COURSE NUMBER : E471

TIME ALLOWED : THREE HOURS

INSTRUCTIONS : ANSWER ANY FOUR OUT OF FIVE QUESTIONS. EACH QUESTION CARRIES 25 MARKS.

MARKS FOR DIFFERENT SECTIONS ARE SHOWN IN THE RIGHT-HAND MARGIN.

STUDENTS ARE PERMITTED TO USE MAPLE TO ANSWER THE QUESTIONS.

THIS PAPER HAS SIX PAGES, INCLUDING THIS PAGE.

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E471 Complex Variables

Question one

(a) Given $w = -7 + 24i$, convert it into polar form and find all the four roots of $\sqrt[4]{w}$ (i.e., the solutions of z in $z^4 = w$). **(8 marks)**

(b) Given $f(z) = e^{-z}\,\bar{z}$ where $\bar{z}$ is the complex conjugate of z, is f(z) an analytical function? Justify your answer. **(10 marks)**

(c) Given the equation $\ln z = -2 - \frac{3}{2}i$, solve for z. **(7 marks)**

Question two

(a) Given $u(x,y) = 5x^3 - 15xy^2 - 3e^{-2x}\cos(2y)$,

(i) show that u(x,y) is a harmonic function , **(4 marks)**

(ii) find its conjugate harmonic function , v(x,y) . **(7 marks)**

(b) Given $f(z) = \dfrac{\sin(z)}{z-3}$, $z_1 = 2i$ and $z_2 = 8$,

(i) find the value of $\int_{z_1, L}^{z_2} f(z)\, dz$ if L is a straight line from z_1 to z_2 , **(11 marks)**

(ii) set $z_3 = -5i$, if L is a straight line from z_1 to z_3 and then joined by a straight line from z_3 to z_2 , is the value of $\int_{z_1, L}^{z_2} f(z)\, dz$ here going to be the same or different from that obtained in (b) (ii) ? Why?

(3 marks)

Question three

(a) Given $f(z) = \dfrac{1}{z^2 + 4}$, find its convergent series representation about the expansion centre of $z_0 = 1 + 2i$ for the region of $1 < |z - z_0| < \sqrt{17}$.

(10 marks)

(b) Given a power series of $\displaystyle\sum_{n=0}^{\infty} \frac{(3n)!}{5^n\,(n!)^3}\, z^n$, find its radius of convergence.

(5 marks)

(c) Given a definite integral of $\displaystyle\int_0^{2\pi} \frac{1 + \sin(\theta)}{25 - 14\cos(2\theta)}\, d\theta$, convert it to a complex contour integral and utilize the residue theorem to find its value. **(10 marks)**

Question four

Convert the following definite integrals into complex contour integrals and utilize the residue theorem to find

(a) the value of $\int_{-\infty}^{\infty} \frac{dx}{\left(x^2 - 3x + 7\right)^2}$ **(7 marks)**

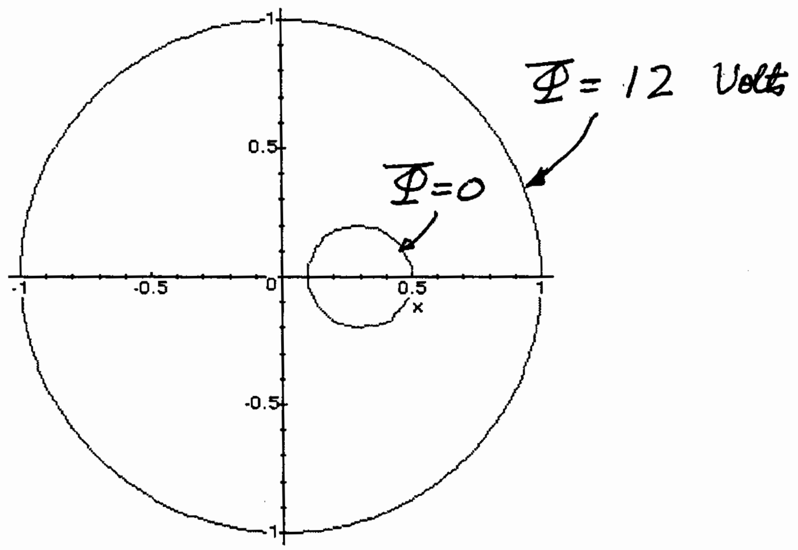
(b) the value of $\int_{-\infty}^{\infty} \frac{\cos(3x)}{x^2 + x + 4} \, dx$ **(9 marks)**

(c) the principal value of $\int_{-\infty}^{\infty} \frac{x}{x^3 + x^2 + x + 1} \, dx$ **(9 marks)**

Question five

(a) Given a linear fractional transformation $w = \dfrac{z-b}{bz-1}$ where $b \neq 0$, show that the unit circle centered at the origin in the z – plane maps to the unit circle centered at the origin in the w – plane . **(6 marks)**

(b) Two long hollow metal cylinders eccentrically located one inside the other have their circular cross sections lying on the x-y plane as shown below :



The outer cylinder has a radius of unity and centered at the origin while the inner one has a radius of 0.2 and centered at (0.3 , 0) .

(i) Find the appropriate value of b given in (a) that can map these two off-centered cylinders in the z-plane to a pair of coaxial cylinders in the w -plane. Find also the radius of the mapped inner cylinder in the w -plane. **(8 marks)**

(ii) If the electric potential of the outer cylinder is 12 volts and that of the inner one is zero, find the potential between the cylinders. Plot the equal potential surfaces of 0 , 4 , 8 and 12 volts in the z-plane. **(11 marks)**