UNIVERSITY OF SWAZILAND

FINAL EXAMINATIONS 2007/2008

B.Sc. / B.Ed. / B.A.S.S.III

TITLE OF PAPER	:	VECTOR ANALYSIS
COURSE NUMBER	:	M312
TIME ALLOWED	:	THREE (3) HOURS
INSTRUCTIONS	:	1. THIS PAPER CONSISTS OF SEVEN QUESTIONS. 2. ANSWER ANY FIVE QUESTIONS
SPECIAL REQUIREMENTS	:	NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

QUESTION 1

(a) (i) Find the scale h_1, h_2, and h_3 in cylindrical and in spherical coordinates. Hence find the volume element dV (in cylindrical and in spherical coordinates. [9]

(ii) Show that the spherical coordinate system is orthogonal. [3]

(b) Let $\mathbf{u}(x,y,z) = x\hat{\mathbf{i}} - y\hat{\mathbf{j}}$ and $\mathbf{v}(x,y,z) = \dfrac{\mathbf{u}}{(x^2+y^2)^{\frac{1}{2}}}$ be vectors in space. Compute the divergence and the curl of $\mathbf{u}$ and $\mathbf{v}$. [8]

QUESTION 2

(a) Integrate $f(x,y,z) = 2x - 6y^2 + 2z$ over the line segment C joining the points (2,2,2) and (3,3,3). [6]

(b) Find the work done in moving a particle in the counterclockwise direction once around the ellipse $\dfrac{x^2}{4} + \dfrac{y^2}{3} = 1$ if the force field is given by $\mathbf{F} = (3x - 4y)\hat{\mathbf{i}} + (4x + 2y)\hat{\mathbf{j}} - 4y^2\hat{\mathbf{k}}$. [8]

(c) Find the circulation of the field $\mathbf{F} = (x - y)\hat{\mathbf{i}} + x\hat{\mathbf{j}}$ around the circle $x^2 + y^2 = 1$. [6]

QUESTION 3

(a) Show that $y\mathrm{d}x + x\mathrm{d}y + 4\mathrm{d}z$ is exact, and evaluate the integral

$$\int_{(1,1,1)}^{(2,3,-1)} y\mathrm{d}x + x\mathrm{d}y + 4\mathrm{d}z.$$

[8]

(b) Find out which of the fields given below are conservative. For conservative fields, find a potential function.

(i) $\mathbf{F} = (yz^2)\hat{\mathbf{i}} + (xz^2)\hat{\mathbf{j}} + (x^2yz)\hat{\mathbf{k}}$.

(ii) $\mathbf{F} = (e^x \sin y)\hat{\mathbf{i}} + (e^x \cos y + \sin z)\hat{\mathbf{j}} + (y \cos z)\hat{\mathbf{k}}$. [12]

QUESTION 4

(a) By any method, find the integral of $g(x, y, z) = xyz$ over the surface of the cube cut from the first octant by the planes $x = 1$, $y = 1$, and $z = 1$. [10]

(b) Give a formula $\mathbf{F} = M(x, y)\hat{\mathbf{i}} + N(x, y)\hat{\mathbf{j}}$ for the vector field in the plane with the properties that $\mathbf{F} = \mathbf{0}$ at the origin and that at any other point (a, b) in the plane, $\mathbf{F}$ is tangent to the circle $x^2 + y^2 = a^2 + b^2$ and points in the clockwise direction, with magnitude $|\mathbf{F}| = \sqrt{a^2 + b^2}$. [10]

QUESTION 5

(a) By any method, find the outward flux of the field $\mathbf{F} = (6x^2 + 2xy)\hat{\mathbf{i}} + (2y + x^2z)\hat{\mathbf{j}} + (4x^2y^3)\hat{\mathbf{k}}$ across the boundary of the region cut from the first octant by the cylinder $x^2 + y^2 = 4$ and the plane $z = 3$. [10]

(b) Verify the divergence theorem for $\mathbf{F} = (2x - z)\hat{\mathbf{i}} + x^2y)\hat{\mathbf{j}} + xz^2\hat{\mathbf{k}}$ over the region bounded by $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$, and $z = 1$. [10]

QUESTION 6

(a) Evaluate:

(i) $\Gamma(6.8)$, given that $\Gamma(1.8) = 0.9314$, [2]

(ii) $\int_0^\infty x^m e^{-a^n} dx$, where m and n are positive integers. [8]

(b) Show that

$$\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta \cos^{2n-1}\theta d\theta = \frac{\Gamma(m)\Gamma(n)}{2\Gamma(m+n)}, \text{ where } m, n > 0.$$ [10]

QUESTION 7

Let $J_n(x)$ be the Bessel function of the first kind of order n.

(a) Prove that

$$J_{-n}(x) = (-1)^n J_n \quad \text{for} \quad n = 1, 2, 3 \ldots.$$

[6]

(b) Use recurrence relations to show that

$$2J_0''(x) + J_0(x) - J_2(x) = 0.$$

[8]

(c) Express $J_4(ax)$ in terms of $J_0(ax)$ and $J_1(ax)$. [6]

END OF EXAMINATION