UNIVERSITY OF SWAZILAND

FINAL EXAMINATIONS 2008

BSc. / BEd. / B.A.S.S. IV

TITLE OF PAPER	:	NUMERICAL ANALYSIS II
COURSE NUMBER	:	M 411
TIME ALLOWED	:	THREE (3) HOURS
INSTRUCTIONS	:	1. THIS PAPER CONSISTS OF SEVEN QUESTIONS. 2. ANSWER ANY FIVE QUESTIONS
SPECIAL REQUIREMENTS	:	NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

QUESTION 1

1. (a) Write the following differential equation as an equivalent system of first order equations. [4 marks]

$$y'' = y'(1 - y^2) - y \qquad \text{subject to} \quad y(0) = 1, \;\; y'(0) = 2.$$

(b) Use finite differences with step size $h = 0.5$ and a centred approximation to approximate the solution of the following differential equation at $x = 0.5$. [6 marks]

$$y'' = -4 \qquad \text{subject to} \quad y(0) = 1 \quad \text{and} \quad y(1) = 1.$$

(c) Use four terms of the Taylor series expansion to solve the following system of differential equations. [10 marks]

$$\begin{aligned} \frac{dx}{dt} &= x(1 - y) \,, \qquad & x(0) = 2 \\ \frac{dy}{dt} &= y(x - 1) \,, \qquad & y(0) = 2 \,. \end{aligned}$$

QUESTION 2

2. Consider the initial value problem

$$\frac{dy}{dx} = 2y^2(x - 1) \;\; \text{with} \;\; y = -\frac{1}{2} \;\; \text{when} \;\; x = 2$$

(a) Use the Taylor expansion of degree 3 for y about $x = 2$ to approximate the value for $y(2.2)$. [8 marks]

(b) Use Euler's method to approximate $y(2.2)$ given that $y(2) = -0.5$ and a step size of $h = 0.1$.

(c) Compare the solution obtained in (a) and (b) against the exact solution of the differential equation. [6 marks]

QUESTION 3

3. (a) Solve the following Poisson equation over the rectangular region $\{(x, y) : \ 0 \le x \le 1.5, \ 0 \le y \le 1\}$ with $h = k = 0.5$.

$$u_{xx} + u_{yy} = 12xy$$

subject to

$$u(x,0) = 0 \ , \quad u(x,1) = 6x; \qquad 0 \le x \le 1.5;$$

$$u(0,y) = 0, \quad u(1.5,y) = 3y^2; \qquad 0 \le y \le 1$$

[10 marks]

(b) Consider the following differential equation system

$$\frac{\partial u}{\partial t} - (1+x)\frac{\partial^2 u}{\partial x^2} = 0, \qquad 0 \le x \le 1, \qquad t > 0$$

$$u(0,t) = 0, \qquad u(1,t) = 0, \qquad t > 0$$

$$u(x,0) = 1 - 2x, \qquad 0 \le x \le 1$$

Use the implicit Backward difference in time and central difference in space (BTCS) scheme with step sizes $h = k = 0.25$ to show that the problem can be written in matrix form as

$$\mathbf{u}_j = A\mathbf{u}_{j+1}$$

where

$$A = \begin{pmatrix} 11 & -5 & 0 \\ -6 & 13 & -6 \\ 0 & -7 & 15 \end{pmatrix}$$

[10 marks]

QUESTION 4

4. Use the 4th-order Runge-Kutta method with $h = 0.1$ to solve the following initial value problem to estimate the values of $y(0.1)$ and $y'(0.1)$;

$$y'' - 2y' + 2y = e^{2x} \sin x$$

subject to

$$y(0) = -0.4 \ , \quad y'(0) = -0.6$$

[20 marks]

QUESTION 5

5. (a) Apply the modified Euler method to solve the following initial value problem

$$y'' = -y^2 + y' + x$$

subject to

$$y(0) = 1 \ , \quad y'(0) = 2$$

with step size $h = 0.1$ to obtain an approximation to $y(0.1)$ and $y'(0.1)$ [10 marks]

(b) For Chebyshev polynomials, prove each of the following statements:

$$(T_i, T_j) = \begin{cases} 0, & i \neq j \\ \pi, & i = j = 0 \\ \frac{\pi}{2}, & i = j > 0 \end{cases}$$

[10 marks]

QUESTION 6

6. (a) Find the second degree least-squares polynomial approximation of the form $P_2(x) = a_o + a_1 x + a_2 x^2$. that best fits the through the following experimental data.

x	0	1	2	4	6
y	3	1	0	1	4

[10 marks]

(b) Discuss consistency, zero-stability and convergence of the linear multi-step method

$$y_{n+2} = 2y_n - y_{n+1} + \frac{h}{2}\{5f_{n+1} + f_n\}.$$

[10 marks]

QUESTION 7

7. Consider the parabolic differential equation

$$\frac{\partial u}{\partial t} - \alpha^2 \frac{\partial^2 u}{\partial x^2} = 0, \quad 0 \le x \le 1, \quad t > 0$$

$$u(0,t) = 0, \quad u(1,t) = 0, \quad t > 0$$

$$u(x,0) = \cos 2\pi x, \quad 0 \le x \le 1$$

If an $O(k^2 + h^2)$ numerical method is constructed using the central difference quotient to approximate u_t and the usual difference quotient to approximate u_{xx},

(a) Write down the finite difference scheme for the problem. [10 marks]

(b) Show that the method has the matrix form

$$\mathbf{u}^{(j+1)} = \mathbf{u}^{(j-1)} + A\mathbf{u}^{(j)} \quad \text{for each} \quad j = 0, 1, 2, \ldots$$

where $\mathbf{u}^{(j)} = (u_{1,j}, u_{2,j}, \ldots, u_{m-1,j})^T$ and A is a tridiagonal matrix. [10 marks]