UNIVERSITY OF SWAZILAND

FINAL EXAMINATION 2008/9

BSc. /BEd. /B.A.S.S IV

TITLE OF PAPER	:	ABSTRACT ALGEBRA II
COURSE NUMBER	:	M 423
TIME ALLOWED	:	THREE (3) HOURS
INSTRUCTIONS	:	1. THIS PAPER CONSISTS OF SEVEN QUESTIONS. 2. ANSWER ANY FIVE QUESTIONS. 3. Non-programmable calculators may be used.
SPECIAL REQUIREMENTS	:	NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

QUESTION 1

1. (a) Give an example of a ring satisfying the following given conditions. (Do not prove anything)
   - (i) A commutative ring with zero divisors.
   - (ii) A division ring that is not a field.
   - (iii) A ring that is not a division ring. [6 marks]

   (b) A ring R is a boolean ring if $a^2 = a, \forall a \in \mathbb{R}$. Show that every boolean ring is commutative. [6 marks]

   (c) Prove that if D is an integral domain, then $D[x]$ is also an integral domain. [8 marks]

QUESTION 2

2. (a) Let R be a commutative ring. Show that $N_a = \{x \in R : ax = 0\}$ is an ideal of R. (Assume the group properties) [4 marks]

   (b) Decide the irreducibility or otherwise of
   - (i) $x^3 - 7x^2 + 3x + 3 \in \mathbb{Q}[x]$.
   - (ii) $8x^3 + 6x^2 - 9x + 24 \in \mathbb{Q}[x]$. [8 marks]

   (c) Show that for a field F, the set of matrices of the form

$$\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \text{ for } a, b \in F$$

   is a right ideal but not a left ideal of $M_2(F)$. [8 marks]

QUESTION 3

3. (a) Describe all units in each of the following rings.
   - (i) $\mathbb{Z}_7$.
   - (ii) $\mathbb{Z} \times \mathbb{Q} \times \mathbb{Z}$. [4 marks]

   (b) Write $x^3 + 3x^2 + 3x + 4 \in \mathbb{Z}_5[x]$ as a product of irreducible polynomials. [6 marks]

   (c) State the Eisentein irreducibility criterion and use it to prove that if p is prime, then the cyclotomic polynomial

$$f(x) = \frac{x^p - 1}{x - 1}$$

   is irreducible. [10 marks]

QUESTION 4

4. (a) Use Fermat's theorem to compute the remainder when 8^{103} is divided by 13. [5 marks]

   (b) Prove that every finite integral domain is a field. [5 marks]

   (c) Show that the mapping $\phi : \mathbb{C} \to \mathbb{R}$ given by

$$(a+ib)\phi = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \quad a, b \in \mathbb{R}$$

   is a ring homorphism. Find its kernel. [6 marks]

   (d) State Kronecker's theorem (Do not prove). [4 marks]

QUESTION 5

5. (a) Show that $1 + x + x^2 \in \mathbb{Z}_2[x]$ is an irreducible polynomial over $\mathbb{Z}_2$. List all the four elements of the field $\mathbb{Z}_2[x]/ < 1 + x + x^2 >$ in the form $a + b\alpha$ with $a, b \in \mathbb{Z}_2$, where α is a root of $1 + x + x^2$ in this extension field of $\mathbb{Z}_2[x]$. Construct Cayle tables for addition and multiplication, showing all computations. [10 marks]

   (b) For each of the given algebraic numbers $\alpha \in \mathbb{C}$ find $\text{irr}(\alpha, \mathbb{Q})$ and $\deg(\mathbb{Q})$.

      (i) $\sqrt{3 - \sqrt{6}}$. [3 marks]

      (ii) $\sqrt{\frac{1}{3} + \sqrt{7}}$. [4 marks]

      (iii) $\sqrt{2} + i$. [3 marks]

QUESTION 6

6. (a) Given that every element β of $E = F(\alpha)$ can be uniquely expressed in the form

$$\beta = b_0 + b_1\alpha + b_2\alpha^2 + \cdots + b_{n-1}\alpha^{n-1}$$

   where each of $b_i \in \mathbb{F}$, α algebraic over the field F and $\deg(\alpha, F) = n \geq 1$. Show that if F is finite with q elements, then $E = F(\alpha)$ has q^n elements. [6 marks]

   (b) Prove that every finite integral domain is a field. [6 marks]

(c) Find the greatest common divisor $d(x)$ of the polynomials $f(x) = x^4 + 4x^3 + 7x^2 + 6x + 2$ and $g(x) = x^3 + 4x^2 + 7x + 4$ over $\mathbb{Q}$ and express $d(x)$ as a linear combination of $f(x)$ and $g(x)$. [8 marks]

QUESTION 7

7. (a) Prove that if R is a ring with unity and N is an ideal of R containing a unit, then $N = R$. [6 marks]

   (b) Find all ideals and maximal ideals $\mathbb{Z}_{18}$. [4 marks]

   (c) In a ring $\mathbb{Z}_n$ show that

      (i) divisors of zero are those elements that are **NOT** relativey prime to n. [5 marks]

      (ii) elements that are relatively prime to n cannot be zero divisors. [5 marks]