UNIVERSITY OF SWAZILAND

107

SUPPLEMENTARY EXAMINATION 2011/2012

BSc. /BEd. /B.A.S.S III

TITLE OF PAPER	:	REAL ANALYSIS
COURSE NUMBER	:	M 331
TIME ALLOWED	:	THREE (3) HOURS
INSTRUCTIONS	:	1. THIS PAPER CONSISTS OF SEVEN QUESTIONS.
		2. ANSWER ANY FIVE QUESTIONS
SPECIAL REQUIREMENTS	:	NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

QUESTION 1

1. (a) Prove that $a^2 < b^2 \iff 0 < a < b$. [5 marks]

   (b) Explain precisely the statement:
   "A set S of real numbers is bounded". [2 marks]

   (c) Determine whether the following sets are bounded or not. Justify your answers.

      i. $S := \{x \in \mathbb{R} : |x+2| = 1 + |3-x|\}$. [4 marks]

      ii. $S := \{\frac{n}{n+1} : n \in \mathbb{N}\}$. [4 marks]

   (d) Let $\alpha > 0$ and let $T := \{\alpha s \in \mathbb{R} : s \in S\}$. Prove that $\inf(T) = \alpha \inf S$. [5 marks]

QUESTION 2

2. (a) Explain precisely the statement "A real number l is the limit of a sequence (x_n) of real numbers. [2 marks]

   (b) Use your definition in 2a to prove the following:

      i. $\lim \left(\dfrac{\sin n}{n^2}\right) = 0$. [4 marks]

      ii. $\lim \left(\dfrac{2n}{n+5}\right) = 2$. [4 marks]

   (c) i. State and prove the squeeze theorem for sequences of real numbers. [6 marks]

      ii. Use the squeeze theorem to prove that $\lim \left(\dfrac{(\sin n)^2}{n+1}\right) = 0$. [4 marks]

QUESTION 3

3. (a) Let $f, g : [a, b] \to \mathbb{R}$ be functions, and let $c \in (a, b)$.

   i. Explain precisely the statement "f is continuous at c". [2 marks]

   ii. Show that the constant function $f(x) \equiv d$ is continuous at c.[4 marks]

   iii. Prove that if both f and g are continuous at c then the sum $f + g$ is also continuous at c. [4 marks]

   iv. Is the converse of 3(a)iii true? Justify your answer. [2 marks]

   (b) State the Intermediate value theorem and use it to show that the equation $\sin x - x = 0$ has a solution in the interval $[0, \pi]$. [5 marks]

   (c) Is the following statement true or false? Justify your answer.
   If the absolute value function $|f| : [0, 1] \to \mathbb{R}$ defined by $|f|(x) := |f(x)|$ is continuous then so is the function $f : [0, 1] \to \mathbb{R}$. [3 marks]

QUESTION 4

4. (a) Let $f : (a, b) \to \mathbb{R}$ be a function.

   i. Explain the statement "f is **not** differentiable at $c \in (a, b)$".[2 marks]

   ii. Show that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = |2x + 1|$ is not differentiable at $x = -\frac{1}{2}$. [4 marks]

   (b) i. State the Mean value theorem for derivatives. [2 marks]

   ii. Use the Mean value theorem for derivatives to prove each of the following statements.

   A. $\frac{2}{3} < \ln 3 < 2$. [5 marks]

   B. Let $f : [a, b] \to \mathbb{R}$ be a function which is both continuous and differentiable on (a, b). If $f'(x) > 0$, $\forall x \in (a, b)$, then f is strictly increasing on (a, b). [5 marks]

QUESTION 5

5. (a) What does it mean to say that a series $\sum a_n$ in $\mathbb{R}$ converges? [2 marks]

(b) Prove that if $\sum a_n, \sum b_n$ converge, then $\sum(a_n + b_n)$ converges. [5 marks]

(c) i. State the Cauchy convergence criterion for series in $\mathbb{R}$. [2 marks]

ii. Prove that the harmonic series $\sum \frac{1}{n}$ diverges. [5 marks]

(d) Do the following series converge or not? Justify your answers.

i. $-1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \ldots.$ [3 marks]

ii. $1 + \frac{1}{2} + \frac{1}{3!} + \frac{1}{4!} + \ldots.$ [3 marks]

QUESTION 6

6. (a) If $f : [a, b] \to \mathbb{R}$ be a function, then explain in detail the statement "f is Reimann integrable on $[a, b]$". [10 marks]

(b) Given that $f(x) := x, \forall x \in [1, 2]$, prove that the function f is Reimann integrable on $[1, 2]$ and find $\int_1^2 x$. [10 marks]

QUESTION 7

7. (a) i. State the infimum property of $\mathbb{R}$. [2 marks]

ii. Let u be a lower bound for a non-empty subset V of $\mathbb{R}$. State a necessary and sufficient condition for u to equal $\inf V$. [2 marks]

iii. Let S and T be non-empty subsets of $\mathbb{R}$. Define
$S + T := \{x + y \in \mathbb{R} : x \in S, y \in T\}$.
Use your result of 7(a)ii above (or otherwise) to show that if both S and T are bounded above then $\inf(S + T) = \inf S + \inf T$. [6 marks]

(b) Determine whether each of the following statements is true or false. Justify your answers.

i. Every sequence of reals numbers that is both bounded and monotone is convergent. [2 marks]

ii. There are two distinct Riemann integrable functions $f, g : [0, 1] \to \mathbb{R}$ such $f < g$ and yet neither $\int_0^1 f > \int_0^1 f$. [2 marks]

iii. $\mathbb{N}$ is bounded in $\mathbb{R}$. [2 marks]

iv. There is a bounded sequence of real numbers which is divergent. [2 marks]

v. Every function $f : [-1, 1] \to \mathbb{R}$ that is continuous on $[-1, 1]$ is also differentiable on $[-1, 1]$. [2 marks]