UNIVERSITY OF SWAZILAND 126

EXAMINATION, 2016/2017

BASS III, B.Ed (Sec.) III, B.Sc. III

Title of Paper : Real Analysis

Course Number : M331

Time Allowed : Three (3) Hours

Instructions

1. This paper consists of SIX (6) questions in TWO sections.
2. Section A is **COMPULSORY** and is worth 40%. Answer ALL questions in this section.
3. Section B consists of FIVE questions, each worth 20%. Answer ANY THREE (3) questions in this section.
4. Show all your working.
5. Start each new major question (A1, B2 – B6) on a new page and clearly indicate the question number at the top of the page.
6. You can answer questions in any order.

Special Requirements: NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

SECTION A [*40 Marks*]: ANSWER ALL QUESTIONS

QUESTION A1 [*40 Marks*]

(a) Consider a set $A \subseteq \mathbb{R}$.

i. Explain what it means to say that the set A is bounded above and define $\sup(A)$ for such a set. (3)

ii. Explain what it means to say that the set A is bounded below and define $\inf(A)$ for such a set. (3)

(b) Let $A = \{x \in \mathbb{R} : x^2 < 2\}$. Find (if they exist) $\min(A)$, $\max(A)$, $\inf(A)$, and $\sup(A)$. Is A bounded? (6)

(c) Give the $\varepsilon - \delta$ definition for $\lim_{x \to a} f(x) = L$, where f is a real-valued function. (3)

(d) Give the $\varepsilon - N$ definition for $\lim_{n \to \infty} x_n = x$ where $\{x_n\}$ is a sequence of real numbers. (3)

(e) Fill in the blanks:

A sequence $\{x_n\}$ diverges to ∞ if for every ______ there is ______ such that ______ whenever ______. (4)

(f) True or False? Explain your answer.

i. *If f is continuous at c, then f is differentiable at c.* (3)

ii. *If f is integrable on $[a, b]$, then f is continuous on $[a, b]$.* (3)

iii. *If $\lim_{n \to \infty} x_n = 0$, then the series $\sum_{n=1}^{\infty} x_n$ is convergent.* (3)

iv. *If the series $\sum_{n=1}^{\infty} x_n$ is convergent, then it converges absolutely.* (3)

(g) State the test for divergence for a series $\sum_{n=1}^{\infty} x_n$. (3)

(h) State Riemann's integrability criterion. (3)

END OF SECTION A – TURN OVER

SECTION B: ANSWER ANY *THREE* QUESTIONS

QUESTION B2 [20 *Marks*]

Test each of the following series for convergence or divergence.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{5n^2+4}$. (4)

(b) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{3^n}$. (4)

(c) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n^2+1}{2n^2+1}\right)^n$. (4)

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$. (8)

QUESTION B3 [20 *Marks*]

(a) Give a precise $\varepsilon - N$ argument to show that

$$\lim_{n\to\infty} \frac{n^2}{2n^2+1} = \tfrac{1}{2}.$$

(7)

(b) i. State the Cauchy convergence criterion for sequences. (3)

ii. Use the Cauchy convergence criterion to show that the sequence

$$\left\{\frac{n+1}{n}\right\}$$

is convergent. (7)

(c) The sequence $x_n = (-1)^n$ does not converge. For what values of $\varepsilon > 0$ is it nonetheless true that there exists a natural number N such that

$$|x_n - 1| < \varepsilon, \quad \forall n \geq N?$$

(3)

QUESTION B4 [20 *Marks*]

(a) i. State the Intermediate Value Theorem. (3)

ii. Show that the equation $x^2 = \cos x$ has a solution in the interval $[0, \pi/2]$. (3)

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Explain what each of the following statements means.

i. f is *right-differentiable* at $c \in [a, b)$. (2)

ii. f is *right-differentiable* at $c \in [a, b)$. (2)

(c) Give an example of a function that is left-differentiable and right-differentiable at a point c in its domain but not differentiable at c. (3)

(c) Let $f(x) = 10x - 11$. Use an $\varepsilon - \delta$ argument to show that $\lim_{x\to 2} f(x) = 9$. (7)

QUESTION B5 [20 *Marks*]

(a) Let $f : [0, 1] \to \mathbb{R}$ be defined by $f(x) = x^3$ and let $P = \{0, 0.1, 0.4, 1\}$. Find $U(P, f)$ and $L(P, f)$. (6)

(b) State the Fundamental Theorem of Calculus. (4)

(c) Let $f : [-1, 1] \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} 0, & \text{if } x \leq 0, \\ 1 & \text{if } x > 0. \end{cases}$$

Show that f is Riemann integrable and find $\int_{-1}^{1} f(x)\, dx$. (10)

QUESTION B6 [20 *Marks*]

(a) Prove or disprove: If $\sum x_n$ and $\sum y_n$ are convergent series, then $\sum x_n y_n$ is convergent. (4)

(b) Prove: *If* $\sum_{n=1}^{\infty} x_n$ *is convergent, then* $\lim_{n\to\infty} x_n = 0$. (8)

(c) Prove: *If the series* $\sum_{n=1}^{\infty} x_n$ *converges absolutely, then it is convergent.* (8)

END OF EXAMINATION PAPER